

# The Class of Purely Unrectifiable Sets in $\ell_2$ is $\Pi_1^1$ -complete

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## Abstract

The space  $F(\ell_2)$  of all closed subsets of  $\ell_2$  is a Polish space. We show that the subset  $P \subset F(\ell_2)$  consisting of the purely 1-unrectifiable sets is  $\Pi_1^1$ -complete.

## 1 Introduction

The concepts of unrectifiable and purely unrectifiable sets are central in contemporary geometric measure theory, see e.g. [Mat99]. In some sense these are sets which are not capturable by smooth approximations: a set is *unrectifiable*, if it cannot be covered (upto a negligible set) by countably many  $C^1$ -curves and 1-*purely unrectifiable*, if its 1-dimensional Hausdorff measure restricted to any  $C^1$ -curve is zero. We only consider 1-purely unrectifiable sets in this article (as opposed to  $m$ -purely unrectifiable for  $m > 1$ ), so we skip the “1” from the notation. There are several open question concerning (partial) characterisations of purely unrectifiable sets such as for example whether or not the two-dimensional Brownian motion is purely unrectifiable with probability 1 [Pre13].

Here we show that the notion of pure unrectifiability is subtle to the extend that any decision procedure for deciding whether a given closed subset of  $\ell_2$  is purely unrectifiable or not requires an exhaustive search through continuum many cases, that is to say, in the language of descriptive set theory, the set of all closed purely unrectifiable subsets of  $\ell_2$  is  $\Pi_1^1$ -hard. On the other hand there *is* a decision procedure of this sort, so the set is  $\Pi_1^1$ -complete (or *coanalytic complete*).

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## 2 Basic Definitions

In order to define purely unrectifiable sets in  $\ell_2$ , let us review the definition of  $C^1$ -curve in  $\ell_2$ :

**1 Definition.** A *Fréchet derivative* of a function  $f: [0, 1] \rightarrow \ell_2$  at point  $x \in [0, 1]$  is a linear operator  $A_x: \mathbb{R} \rightarrow \ell_2$  such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - A_x h\|_2}{|h|} = 0.$$

The function belongs to  $C^1$ , if the Fréchet derivative exists at every point and the map  $x \mapsto A_x$  is continuous in the operator norm.

The linear operator  $A_x$  is uniquely determined by the vector  $A_x(1)$ , so denote  $f'(x) = A_x(1)$ . Also denote the space of all  $C^1$ -curves by  $C^1([0, 1], \ell_2)$ .

**2 Definition.** A subset  $N$  of  $\ell_2$  is *purely unrectifiable*, if it is null on every  $C^1$ -curve. That is, given a  $C^1$ -map  $f: [0, 1] \rightarrow \ell_2$ , the one-dimensional Hausdorff measure of  $N \cap \text{ran}(f)$ , denoted  $\mathcal{H}^1(N \cap \text{ran}(f))$ , equals 0. Denote the set of purely unrectifiable curves in  $\ell_2$  by  $P$ .

## 3 Preliminaries in Descriptive Set Theory

We follow the notation and presentation of the book “Classical Descriptive Set Theory” by A. Kechris [Kec94] and refer frequently to it below when addressing well-known facts.

A *Polish space* is a separable topological space which is homeomorphic to a complete metric space. The Hilbert space  $\ell_2$  is an example of a Polish space. A *standard Borel space* is a set  $X$  endowed with a  $\sigma$ -algebra  $S$  such that there exists a Polish topology on  $X$  in which the Borel sets are precisely the sets in  $S$ .

Let  $F(\ell_2)$  denote the set of all closed subsets of  $\ell_2$ . This is a standard Borel space where the  $\sigma$ -algebra is generated by the sets of the form

$$\{A \in F(\ell_2) \mid A \cap U \neq \emptyset\} \tag{B}$$

where  $U$  ranges over the basic open sets of  $\ell_2$  [Kec94, Thm 12.6]. We need the following fact. Let  $H$  be the Hilbert cube  $H = [0, 1]^{\mathbb{N}}$ . By [Kec94, Thm 4.14],  $\ell_2$  can be embedded into  $H$  so that the image is a  $G_\delta$  subset. Let  $e$  be that embedding. Let  $K(H)$  be the set of all compact non-empty subsets of  $H$  equipped with the Hausdorff metric;  $K(H)$  is a compact Polish space.

**3 Fact.** *The embedding  $e: \ell_2 \rightarrow H$  induces an embedding of  $F(\ell_2)$  into  $K(H)$  such that the image of  $F(\ell_2)$  is  $G_\delta$  in  $K(H)$  thus inducing a Polish topology on  $F(\ell_2)$  [Kec94, Thm 3.17]. This topology gives rise to the same Borel sets as (B) above.  $\square$*

By  $\omega$  and by  $\mathbb{N}$  we denote the set of natural numbers, by  $\mathbb{N}_+$  the set of positive natural numbers. For  $n \in \mathbb{N}$ ,  $\omega^n$  is the set of all functions from  $\{0, \dots, n-1\}$

to  $\omega$ ,  $\omega^{<\omega} = \bigcup_{n \in \mathbb{N}} \omega^n$  and  $\omega^\omega$  denotes the set of all functions from  $\omega$  to  $\omega$ . Similarly  $2^\omega$  denotes the set of all functions from  $\omega$  to  $\{0, 1\}$  and  $2^{<\omega}$  the set of functions from  $\{0, \dots, n-1\}$  to  $\{0, 1\}$  for all  $n$ . The spaces  $\omega^\omega$  and  $2^\omega$  are Polish spaces in the product topology.

The set  $\omega^{<\omega}$  can be ordered in a natural way:  $p < q$  if  $q \upharpoonright \text{dom } p = p$ . This is an example of a *tree*. The set of all trees,  $\text{Tr}$  is the set of all downward closed suborderings of  $\omega^{<\omega}$ . The space  $\text{Tr}$  can be endowed naturally with a Polish topology as a closed subset of  $2^{\omega^{<\omega}}$  which is in turn homeomorphic to  $2^\omega$  via a bijection  $\omega \rightarrow \omega^{<\omega}$ . A *branch* of a tree  $T \in \text{Tr}$  is a sequence  $(p_n)_{n < \omega}$  such that  $p_n \in \omega^n$ ,  $p_n < p_{n+1}$  and  $p_n \in T$  for all  $n$ .

A subset of a Polish space  $A \subset X$  is  $\Sigma_1^1$ , if there is a Polish space  $Y$  and a Borel subset  $B \subset X \times Y$  such that  $A$  is the projection of  $B$  to  $X$ . A set is  $\Pi_1^1$  if it is the complement of a  $\Sigma_1^1$  set.

**4 Definition.** A set  $A \subset X$  is *Borel Wadge-reducible* to another  $B \subset Y$  ( $X$  and  $Y$  are Polish), if there exists a Borel function  $f: X \rightarrow Y$  such that for all  $x \in X$ ,  $x \in A \iff f(x) \in B$ . We denote this by  $A \leq_W B$ .

A set  $A \subset X$  is  $\Pi_1^1$ -*hard*, if every  $\Pi_1^1$  set  $B$  is Wadge-reducible to it,  $B \leq_W A$ . Similarly  $\Sigma_1^1$ -*hard*. A set is  $\Pi_1^1$ -*complete* ( $\Sigma_1^1$ -*complete*), if it is  $\Pi_1^1$  and  $\Pi_1^1$ -*hard* ( $\Sigma_1^1$  and  $\Sigma_1^1$ -*hard*).

Since the classes  $\Sigma_1^1$  and  $\Pi_1^1$  are closed under preimages in Borel maps [Kec94, Thm 14.4], it is clear that if  $A$  is  $\Sigma_1^1$  and  $B \leq_W A$ , then  $B$  is also  $\Sigma_1^1$ . On the other hand a simple diagonalisation argument together with the Souslin's Theorem [Kec94, Thm 14.11] shows that there are  $\Pi_1^1$  sets that are not  $\Sigma_1^1$ . Therefore a  $\Pi_1^1$ -hard set cannot be  $\Sigma_1^1$ , because it Wadge reduces some  $\Pi_1^1$  set that is not  $\Sigma_1^1$ . In particular it cannot be Borel.

An example of a  $\Pi_1^1$ -complete set is the set of those trees in  $\text{Tr}$  which do not have a branch [Kec94, 27.1]. To sum up, the main conclusions in this paper are based on the following two facts:

- 5 Fact.**
1. If  $A$  is  $\Pi_1^1$ -hard and  $A \leq_W B$ , then  $B$  is  $\Pi_1^1$ -hard.
  2. The set  $\{T \in \text{Tr} \mid T \text{ has no branches}\}$  is  $\Pi_1^1$ -hard. [Kec94, p. 209] □

## 4 Main Theorem

**6 Proposition.** The set  $P = \{A \in F(\ell_2) \mid A \text{ is purely unrectifiable}\}$  is  $\Pi_1^1$ .

*Proof.* The space  $C^1(\ell_2)$  is Polish in the topology given by the sup-norm. Let  $A \subset F(\ell_2) \times C^1(\ell_2)$  be the set of those pairs  $(C, \gamma)$  such that  $\mathcal{H}^1(C \cap \text{ran } \gamma) > 0$ . Then the projection of  $A$  to the first coordinate is precisely the complement of  $P$ , so it remains to show that  $A$  is Borel.

Fix a dense countable subset  $D$  of  $\ell_2$  and define a basic open set of  $\ell_2$  to be an open ball  $B(x, r)$  where  $r \in \mathbb{Q}$  and  $x \in D$ . Clearly this is a countable basis.

Since  $C \cap \text{ran } \gamma$  is compact, the inequality  $H^1(C \cap \text{ran } \gamma) > 0$  is equivalent to the statement that there exists  $n \in \mathbb{N}$  such that for all finite sequences

$(B(x_1, r_1), \dots, B(x_k, r_k))$  of basic open sets of  $\ell_2$ , if  $\sum_{i=1}^k r_i < 1/n$ , then  $C \cap \text{ran } \gamma \not\subset \bigcup_{i=1}^k B(x_i, r_i)$ . Denoting

$$A^*(x_1, \dots, x_k, r_1, \dots, r_k) = \{(C, \gamma) \in F(\ell_2) \times C^1(\ell_2) \mid C \cap \text{ran } \gamma \not\subset \overline{\bigcup_{i=1}^k B(x_i, r_i)}\},$$

we get

$$A = \bigcup_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcap_{\substack{\bar{x} \in D^k, \bar{r} \in \mathbb{Q}^k \\ r_1 + \dots + r_k < 1/n}} A^*(x_1, \dots, x_k, r_1, \dots, r_k)$$

Being a subset of a closed set is Borel, so  $A^*(x_1, \dots, x_k, r_1, \dots, r_k)$  is Borel. Hence  $A$  is Borel.  $\square$

**7 Theorem** (Main Theorem). *The set  $P = \{A \in F(\ell_2) \mid A \text{ is purely unrectifiable}\}$  is  $\Pi_1^1$ -complete.*

*Proof of Theorem 7.* We already showed (Theorem 6) that  $P$  is  $\Pi_1^1$ , so we want to show that it is  $\Pi_1^1$ -hard. The proof is reminiscent of the proof of [Kec94, Thm 27.6, pp. 210–211].

We will show that the set  $NB$  of those trees  $T \in \text{Tr}$  which do not have a branch is Wadge-reducible to  $P$ . That is, we will find a Borel function  $H: \text{Tr} \rightarrow F(\ell_2)$  such that  $H(T)$  is not purely unrectifiable if and only if  $T$  has a branch. The result follows then from Fact 5.

A Cantor set  $C \subset \mathbb{R}$  with a positive Lebesgue measure can be constructed by removing an open interval of length  $1/4$  from the middle of the closed unit interval  $[0, 1]$ , then removing open intervals of length  $1/16$  from the middle of each of the remaining intervals and so on. At the  $n$ :th step we have a disjoint union of  $2^n$  closed intervals. From left to right, label these intervals by  $C_n^1, \dots, C_n^{2^n}$  and set  $C = \bigcap_{n=0}^{\infty} \bigcup_{k=1}^{2^n} C_n^k$ .

Let  $\{e_{n,m} \mid n, m \in \mathbb{N}\}$  be a basis for  $\ell_2$ . For each  $s \in \omega^{<\omega}$  let us define a finite subset  $v_s$  of  $\ell_2$  as follows:

$$v_s = \left\{ \sum_{n=0}^{\text{dom}(s)-1} \frac{1+p(n)}{\sqrt{2^n}} e_{n,s(n)} \mid p \in 2^{\text{dom}(s)} \right\}.$$

Then for every tree  $T \in \text{Tr}$ , let

$$H(T) = \overline{\bigcup_{s \in T} v_s}.$$

**7.1 Claim.** If  $T \in \text{Tr}$  has a branch, then there is a  $C^1$ -function  $f: [0, 1] \rightarrow \ell_2$  such that the one-dimensional Hausdorff measure of  $H(T) \cap \text{ran } f$  is positive.

*Proof of Claim 7.1.* Suppose that  $T$  has a branch and that  $b \in \omega^\omega$  is such that  $b \upharpoonright n \in T$  for all  $n$ . Let us construct a  $C^1$ -function  $f: [0, 1] \rightarrow \ell_2$  as follows. For  $n \in \mathbb{N}$  define  $f_n: [0, 1] \rightarrow \mathbb{R}$  to be a smooth function such that

- $f_n(x) = \frac{1}{\sqrt{2^n}}$  for  $x \in C_n^k$  when  $k$  is odd, and  $f_n(x) = \frac{2}{\sqrt{2^n}}$  for  $x \in C_n^k$  when  $k$  is even.
- range of  $f_n$  is  $\left[\frac{1}{\sqrt{2^n}}, \frac{2}{\sqrt{2^n}}\right]$
- if  $I$  is an open interval which is removed at the  $k$ :th stage in the construction of  $C$ , and  $x \in I$ , then

$$0 < f'_n(x) \leq \frac{4^{k+1}}{\sqrt{2^n}}.$$

The derivative can be bounded in this way because if  $I$  is an open interval that is removed at the  $k$ :th stage, then  $|I| = 4^{-k}$  and in this interval, the function is only required to either raise from  $1/\sqrt{2^n}$  to  $2/\sqrt{2^n}$  or decrease the same amount in the opposite direction. On the other hand, if  $x \in C$ , then the derivative of  $f_k$  is 0 for all  $k$ .

Now let  $f(x) = \sum_{n=0}^{\infty} f_n(x)e_{n,b(n)}$ . Clearly  $f(x) \in \ell_2$  for all  $x$ :

$$\begin{aligned} \|f(x)\|_2^2 &= \sum_{n=0}^{\infty} |f_n(x)|^2 \\ &\leq \sum_{n=0}^{\infty} \left|\frac{2}{\sqrt{2^n}}\right|^2 \\ &= \sum_{n=0}^{\infty} \frac{2}{2^n} \\ &= 4. \end{aligned}$$

**7.1.1 Subclaim.** The function  $f$  has a Fréchet derivative at each  $x \in [0, 1]$ .

*Proof of Subclaim 7.1.1.* The vector  $A_x = \sum_{n=0}^{\infty} f'_n(x)e_{n,b(n)}$  is in  $\ell_2$ , because the absolute value of  $f'_n(x)$  is bounded by  $\frac{4^{k+1}}{\sqrt{2^n}}$  where  $k$  is a constant natural number which depends on  $x$ . Thus,  $A_x$  defines a bounded linear operator  $h \mapsto A_x h$ . We claim that  $A_x$  is the Fréchet derivative of  $f$  at  $x$ . For that we need to show that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - A_x h\|_2}{|h|} = 0.$$

So assume that  $\varepsilon > 0$ . The numerator can be rewritten as

$$\sqrt{\sum_{n=0}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2}.$$

Let us show first that there exists  $k \in \mathbb{N}$  such that for all  $h$

$$\sum_{n=k}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2 \leq \varepsilon^2 h^2 :$$

$$\begin{aligned}
|f_n(x+h) - f_n(x) - f'_n(x)h|^2 &\leq (|f_n(x+h) - f_n(x)| + |f'_n(x)h|)^2 \\
(\text{mean value theorem}) &= (|f'_n(\xi)||h| + |f'_n(x)||h|)^2 \\
&= (|f'_n(\xi)| + |f'_n(x)|)^2 h^2 \\
(\text{for some constant } K) &\leq \left(\frac{K}{2^n}\right)^2 h^2.
\end{aligned}$$

The last inequality follows from the definition of  $f$ . Therefore for each  $i \in \mathbb{N}$  we have

$$\sum_{n=i}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2 \leq \sum_{n=i}^{\infty} \left(\frac{K}{2^n}\right)^2 h^2.$$

Now, by choosing  $k$  big enough we can make sure that  $\sum_{n=k}^{\infty} \left(\frac{K}{2^n}\right)^2 < \varepsilon^2$ , so pick this  $k$ . Then, for each  $n < k$ , let  $h_n > 0$  be small enough real number such that  $|f_n(x+h_n) - f_n(x) - f'_n(x)h_n| \leq \frac{\varepsilon}{2^n} h_n$  and let  $h = h_\varepsilon = \min_{n < k} h_n$ . Then we have:

$$\begin{aligned}
\frac{\|f(x+h) - f(x) - A_x h\|_2}{|h|} &= \frac{\sqrt{\sum_{n=0}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2}}{|h|} \\
&\leq \frac{\sqrt{\left(\sum_{n=0}^{k-1} |f_n(x+h) - f_n(x) - f'_n(x)h|^2\right) + \varepsilon^2 h^2}}{|h|} \\
&\leq \frac{\sqrt{\left(\sum_{n=0}^{k-1} \left(\frac{\varepsilon}{2^n} h\right)^2\right) + \varepsilon^2 h^2}}{|h|} \\
&< \frac{\sqrt{4\varepsilon^2 h^2 + \varepsilon^2 h^2}}{|h|} \\
&= \sqrt{5}\varepsilon.
\end{aligned}$$

□ Subclaim 7.1.1

**7.1.2 Subclaim.** The Fréchet derivative of  $f$  is continuous. Thus  $f \in C^1([0, 1], \ell_2)$ .

*Proof of Subclaim 7.1.2.* Let  $x \in [0, 1]$  and  $\varepsilon > 0$ . Denote by  $A_x$  the Fréchet derivative of  $f$  at  $x$ , which has the following form by the previous proof:

$$A_x = \sum_{n=0}^{\infty} f'_n(x) e_{n, b(n)}.$$

The norm of a linear operator from  $\mathbb{R}$  to  $\ell_2$  (such as  $A_x$ ) is determined by the norm of the value at 1, thus for example

$$\|A_x\| = \|A_x(1)\|_2 = \sum_{n=0}^{\infty} |f'_n(x)|^2.$$

So for every  $y \in [0, 1]$  we have:

$$\begin{aligned}\|A_x - A_y\| &= \left\| \sum_{n=0}^{\infty} (f'_n(x) - f'_n(y)) e_{n,b(n)} \right\|_2 \\ &= \sqrt{\sum_{n=0}^{\infty} |f'_n(x) - f'_n(y)|^2}.\end{aligned}$$

Now similarly as in the previous proof, let us find  $k \in \mathbb{N}$  such that

$$\sum_{n=k}^{\infty} |f'_n(x) - f'_n(y)|^2 < \varepsilon^2.$$

But

$$|f'_n(x) - f'_n(y)|^2 \leq (|f'_n(x)| + |f'_n(y)|)^2 \leq \left(\frac{K}{2^n}\right)^2,$$

where  $K$  is some constant (this follows again from the definition of  $f$ ). So we can find a big enough  $k$  as required. Now, for every  $i < k$  pick  $\delta_i$  such that for every  $y$  in the  $\delta_i$ -neighbourhood of  $x$  we have  $|f'_n(x) - f'_n(y)| < \varepsilon/2^n$ . This is possible since  $f_n$  are smooth by definition. Then let  $\delta = \min_{i < k} \delta_i$ . Now, if  $y$  is the  $\delta$ -neighbourhood of  $x$ , then by applying the above, we have

$$\begin{aligned}\|A_x - A_y\| &= \sqrt{\sum_{n=0}^{\infty} |f'_n(x) - f'_n(y)|^2} \\ &= \sqrt{\left(\sum_{n=0}^{k-1} |f'_n(x) - f'_n(y)|^2\right) + \sum_{n=k}^{\infty} |f'_n(x) - f'_n(y)|^2} \\ &\leq \sqrt{\left(\sum_{n=0}^{k-1} |f'_n(x) - f'_n(y)|^2\right) + \varepsilon^2} \\ &\leq \sqrt{\left(\sum_{n=0}^{k-1} (\varepsilon/2^n)^2\right) + \varepsilon^2} \\ &< \sqrt{2\varepsilon^2 + \varepsilon^2} \\ &= \sqrt{3}\varepsilon.\end{aligned}$$

□ Subclaim 7.1.2

**7.1.3 Subclaim.**  $f$  is a homeomorphism onto its image.

*Proof of Subclaim 7.1.3.* Since  $\text{dom } f$  is compact, it is sufficient to show that it is injective. Let  $x, y \in [0, 1]$ . If there is an interval  $I$  which is removed at some

stage  $n$  in the construction of  $C$  such that  $x, y \in I$ , then  $f_n(x) \neq f_n(y)$ , because  $f'_n(z) > 0$  for all  $z \in I$  by the definition of  $f_n$ . If not, then find the least stage  $m$  and an interval  $I$  such that  $I$  is removed at the  $m$ :th stage and  $I$  is between  $x$  and  $y$ . Then clearly again  $f_m(x) \neq f_m(y)$ .  $\square$  Subclaim 7.1.3

**7.1.4 Subclaim.**  $(f \upharpoonright C)^{-1}$  is Lipschitz.

*Proof of Subclaim 7.1.4.* If  $\eta \in 2^\omega$ , denote by  $g(\eta)$  the unique point in  $C$  which is obtained by going “left” at stage  $n$  if  $\eta(n) = 0$  and “right” if  $\eta(n) = 1$ . That is,  $g$  is the canonical homeomorphism of  $2^\omega$  onto  $C$ . It is not hard to see that

$$g(\eta) = \sum_{n=0}^{\infty} \eta(n) \frac{2^{n+1} + 6}{4^{n+1}}.$$

Now  $f_n(g(\eta))$  is the image of  $g(\eta)$  under  $f_n$  and by the definition of  $f_n$  we have  $f_n(g(\eta)) = 1/\sqrt{2^n}$  if  $\eta(n) = 0$  and  $f_n(g(\eta)) = 2/\sqrt{2^n}$  if  $\eta(n) = 1$ ; that is  $f_n(g(\eta)) = (1 + \eta(n))/\sqrt{2^n}$ . Let  $\eta$  and  $\xi$  be two arbitrary elements of  $2^\omega$ , thus corresponding to the two (arbitrary) elements  $g(\eta)$  and  $g(\xi)$  of  $C$ . Denote  $c_n = |\eta(n) - \xi(n)|$ . Note that for all  $n \in \mathbb{N}$ ,  $c_n^2 = c_n$ . Then

$$\begin{aligned} d(g(\eta), g(\xi)) &= \left| \sum_{n=0}^{\infty} \eta(n) \frac{2^{n+1} + 6}{4^{n+1}} - \sum_{n=0}^{\infty} \xi(n) \frac{2^{n+1} + 6}{4^{n+1}} \right| \\ &= \left| \sum_{n=0}^{\infty} (\eta(n) - \xi(n)) \frac{2^{n+1} + 6}{4^{n+1}} \right| \\ &\leq \left| \sum_{n=0}^{\infty} |\eta(n) - \xi(n)| \frac{2^{n+1} + 6}{4^{n+1}} \right| \\ &= \sum_{n=0}^{\infty} c_n \frac{2^{n+1} + 6}{4^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{2^n}} \cdot \frac{2^{n+1} + 6}{2^{n+1} \sqrt{2^{n+1}}} \\ (\text{H\"older}) &\leq \sqrt{\sum_{n=0}^{\infty} \frac{c_n}{2^n}} \cdot \underbrace{\sqrt{\sum_{n=0}^{\infty} \frac{2^{n+1} + 6}{2^{n+1} \sqrt{2^{n+1}}}}}_{=:L} \\ &= L \cdot \sqrt{\sum_{n=0}^{\infty} \frac{c_n}{2^n}} \\ &= L \cdot \sqrt{\sum_{n=0}^{\infty} \left| \frac{\eta(n)}{\sqrt{2^n}} - \frac{\xi(n)}{\sqrt{2^n}} \right|^2} \end{aligned}$$



$$\begin{aligned}
&= L \cdot \sqrt{\sum_{n=0}^{\infty} \left| \frac{1 + \eta(n)}{\sqrt{2^n}} - \frac{1 + \xi(n)}{\sqrt{2^n}} \right|^2} \\
&= L \cdot \sqrt{\sum_{n=0}^{\infty} \left| f_n(g(\eta)) - f_n(g(\xi)) \right|^2} \\
&= L \cdot \|f(g(\eta)) - f(g(\xi))\|_2.
\end{aligned}$$

This verifies that the function  $(f \upharpoonright C)^{-1}$  is Lipschitz.

□ Subclaim 7.1.4

Since  $C$  has positive measure, this implies that the one-dimensional Hausdorff measure of  $f[C] = ((f \upharpoonright C)^{-1})^{-1}C$  must also have positive measure. So it remains to show that  $f[C] \subset H(T)$  and the proof of Claim 7.1 is done.

**7.1.5 Subclaim.**  $f[C] \subset H(T)$ .

*Proof of Subclaim 7.1.5.* Suppose  $\eta \in 2^\omega$  and let  $g(\eta)$  be as in the previous proof, the canonical image of  $\eta$  in  $C$ . Then, as above,  $f_n(g(\eta)) = (1 + \eta(n))/\sqrt{2^n}$ , so

$$f(g(\eta)) = \sum_{n=0}^{\infty} \frac{1 + \eta(n)}{\sqrt{2^n}} e_{n,b(n)}.$$

Now, by looking at the definition of  $v_s$ , one can see that the approximations of  $f(g(\eta))$  of the form

$$\sum_{n=0}^{k-1} \frac{1 + \eta(n)}{\sqrt{2^n}} e_{n,b(n)}$$

appear in  $v_{b \upharpoonright k}$ , so  $f(g(\eta)) \in \overline{\bigcup_{s \in T} v_s} = H(T)$ .

□ Subclaim 7.1.5

□ Claim 7.1

**7.2 Claim.** If  $T$  does not have a branch, then  $H(T)$  is countable.

*Proof of Claim 7.2.* If  $H(T)$  is uncountable, then, because  $\bigcup_{s \in T} v_s$  is countable, there is a point  $x$  in  $\overline{\bigcup_{s \in T} v_s} \setminus \bigcup_{s \in T} v_s$ . Let  $(p_i)_{i \in \mathbb{N}}$  be a Cauchy sequence of elements of  $\bigcup_{s \in T} v_s$  converging to  $x$ . By going to a subsequence, we can assume that for all  $i \in \mathbb{N}$ ,  $d(p_{i+1}, p_i) < 2^{-i}$ . The latter inequality implies by the definition of the sets  $v_s$  that if  $\text{dom } s \leq i$ , then

$$p_i \upharpoonright \text{dom } s \in v_s \iff p_{i+1} \upharpoonright \text{dom } s \in v_s.$$

So, we can find  $b \in \omega^\omega$  such that  $p_i \in v_{b \upharpoonright i}$  for all  $i$  and so  $(b \upharpoonright n)_{n \in \mathbb{N}}$  must be a branch in  $T$ .

□ Claim 7.2

By Claims 7.1 and 7.2,  $T$  has no branch if and only if  $H(T)$  is purely unrecifiable which concludes the proof.

□ Theorem 7

## References

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